*On Metallic Reflection and the Influence of the Layer of Transition.**

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(Communicated by Professor J. Larmor, Sec. R.S. Received January 11,—Read January 25, 1906.)

It is well known that when light is propagated in an absorbing medium, the dynamical equations and the boundary conditions are of exactly the same form as for a transparent medium. From a mathematical point of view the only difference between the two cases is that μ, the refractive index in a transparent medium, is replaced in the absorbing medium by a complex quantity $\mu - ia$, where μ is the "refractive index" of the medium, *i.e.*, the ratio of the velocity of light in air to that in the medium, and a is the coefficient of absorption.

When dealing with the problem of reflection we shall take the plane of xy as that of incidence, and $x = 0$ as the surface of separation of the two media; the vectors representing the displacements will then be of the forms $e^{ipt-i(x\cos\phi+y\sin\phi)/V}$ in the incident, and $re^{ipt+i(x\cos\phi-y\sin\phi)/V}$ in the reflected wave. Here ϕ is the angle of incidence for the frequency, and V the velocity of propagation in the first medium. The incident wave is of unit amplitude, and if $r = Re^{i\theta}$, then R and θ represent the amplitude and change of phase in the reflected wave.

When the incident light is polarised perpendicularly to the plane of incidence, r is given by Fresnel's formula

$$r_1 = \frac{\mu\cos\phi - \cos\phi'}{\mu\cos\phi + \cos\phi'} = \frac{\tan(\phi-\phi')}{\tan(\phi+\phi')},$$

where ϕ' is the angle of refraction. For light polarised parallel to the plane of incidence we have the corresponding formula

$$r_2 = \frac{\cos\phi - \mu\cos\phi'}{\cos\phi + \mu\cos\phi'} = \frac{-\sin(\phi-\phi')}{\sin(\phi+\phi')}.$$

When applying these formulæ to the problem of metallic reflection, we have to replace μ by the complex $\mu - ia = Me^{-i\alpha}$. Then ϕ' becomes complex,

[* With regard to the paper "On Newton's Rings formed by Metallic Reflection" ('Roy. Soc. Proc.,' A, vol. 76, 1905, p. 515), of which the proofs were corrected in England, in order to avoid delay, Professor Maclaurin writes:—"I have in one place inadvertently put μ_0 instead of the complex $\mu\,(=\mu_0 - ia)$. Unfortunately this affects some of the results that follow. The errors thus introduced are not numerous, and they do not at all affect the general trend of the argument."]

$\sin\phi=(\mu-ia)\sin\phi'$, and $\cos\phi'=ce^{-iu}=(1-M^{-2}\sin^2\phi\,.\,e^{2ia})^{\frac{1}{2}}$. With this notation we get

$$r_1=R_1e^{i\theta_1}=\frac{M\cos\phi\,.\,e^{-i(\alpha-u)}-c}{M\cos\phi\,.\,e^{-i(\alpha-u)}+c};$$

i. e.,
$$R_1e^{i\theta_1}=\frac{M\cos\phi\cos(\alpha-u)-c-iM\cos\phi\sin(\alpha-u)}{M\cos\phi\cos(\alpha-u)+c-iM\cos\phi\sin(\alpha-u)}.$$

Whence we get

$$R_1^2=\frac{M^2\cos^2\phi+c^2-2Mc\cos\phi\cos(\alpha-u)}{M^2\cos^2\phi+c^2+2Mc\cos\phi\cos(\alpha-u)}=\frac{1-x_1}{1+x_1},$$

where
$$x_1=\frac{2Mc\cos\phi\cos(\alpha-u)}{M^2\cos^2\phi+c^2},$$

and
$$\tan\theta_1=\frac{2Mc\cos\phi\sin(\alpha-u)}{M^2\cos^2\phi-c^2}.$$

Similarly,

$$r_2=R_2e^{i\theta_2}=\frac{\cos\phi-Mce^{-i[\alpha+u]}}{\cos\phi+Mce^{-i[\alpha+u]}}=\frac{\cos\phi-Mc\cos(\alpha+u)+iMc\sin(\alpha+u)}{\cos\phi+Mc\cos(\alpha+u)-iMc\sin(\alpha+u)},$$

whence

$$R_2^2=\frac{M^2c^2+\cos^2\phi-2Mc\cos\phi\cos(\alpha+u)}{M^2c^2+\cos^2\phi+2Mc\cos\phi\cos(\alpha+u)}=\frac{1-x_2}{1+x_2},$$

where
$$x_2=\frac{2Mc\cos\phi\cos(\alpha+u)}{M^2c^2+\cos^2\phi}$$

and
$$\tan\theta_2=\frac{2Mc\cos\phi\sin(\alpha+u)}{M^2c^2-\cos^2\phi}.$$

If M and α be given, these equations suffice to determine R_1, R_2, θ_1, and θ_2 completely.

For some purposes we are mainly interested in the ratio $R_1:R_2$, and in the difference of phase $\theta_1-\theta_2$ between the light polarised perpendicularly and parallel to the plane of incidence. Now, from the above, we have

$$\frac{R_1}{R_2}e^{i(\theta_1-\theta_2)}=\frac{r_1}{r_2}=-\frac{\tan(\phi-\phi')\sin(\phi+\phi')}{\tan(\phi+\phi')\sin(\phi-\phi')}$$
$$=-\frac{\cos(\phi+\phi')}{\cos(\phi-\phi')}=\frac{\sin\phi\sin\phi'-\cos\phi\cos\phi'}{\sin\phi\sin\phi'+\cos\phi\cos\phi'}$$
$$=\frac{\sin^2\phi-Mc\cos\phi e^{-i(\alpha+u)}}{\sin^2\phi+Mc\cos\phi e^{-i(\alpha+u)}}.$$

Thus

$$\left(\frac{R_1}{R_2}\right)^2=\frac{M^2c^2\cos^2\phi+\sin^4\phi-2Mc\cos\phi\sin^2\phi\cos(\alpha+u)}{M^2c^2\cos^2\phi+\sin^4\phi+2Mc\cos\phi\sin^2\phi\cos(\alpha+u)}=\frac{1-x}{1+x},$$

where
$$x=\frac{2Mc\cos\phi\sin^2\phi\cos(\alpha+u)}{M^2c^2\cos^2\phi+\sin^4\phi},$$

and
$$\tan(\theta_2-\theta_1) = \frac{2Mc\cos\phi\sin^2\phi\sin(\alpha+u)}{M^2c^2\cos^2\phi-\sin^4\phi}.$$

The last equation shows that as ϕ increases from 0 to $\frac{1}{2}\pi$, $\theta_2-\theta_1$, increases from 0 to π. We have $\theta_2-\theta_1=\frac{1}{2}\pi$, when

$$M^2c^2\cos^2\phi = \sin^4\phi,$$

and this equation accordingly determines the *Principal Incidence.*

For this angle we have

$$\frac{R_1^2}{R_2^2} = \frac{1-\cos(\alpha+u)}{1+\cos(\alpha+u)} = \tan^2\tfrac{1}{2}(\alpha+u) = \tan^2\beta,$$

where $\beta = \frac{1}{2}(\alpha+u)$ and is the *Principal Azimuth.*

These are substantially the formulæ obtained by Cauchy when discussing the problem of metallic reflection. Before putting them to the test of modern experiments, we shall make some transformations that will be useful for some purposes.

We know from the experimental investigations of Drude and others as to the optical constants of metals, that M^2 is always large. From Drude's results it is least for copper, where its value is 7·27, and greatest for zinc, where it is 34·52. This enables us to expand some of the above functions in ascending powers of $1/M^2$, and so obtain approximate formulæ sufficiently accurate for many purposes.

We have

$$ce^{-iu} = \cos\phi' = (1-M^{-2}\sin^2\phi\,.\,e^{2i\alpha})^{\frac{1}{2}} = 1-\frac{\sin^2\phi\,.\,e^{2i\alpha}}{2M^2}-\frac{\sin^4\phi\,.\,e^{4i\alpha}}{8M^4}-\frac{\sin^6\phi\,.\,e^{6i\alpha}}{16M^6}+\dots.$$

If we multiply each side of this identity by $e^{i\alpha}$ and by $e^{-i\alpha}$, and equate real parts, we get

$$c\cos(\alpha-u) = \cos\alpha-\frac{\sin^2\phi}{2M^2}\cos 3\alpha-\frac{\sin^4\phi}{8M^4}\cos 5\alpha+\dots,$$

$$c\cos(\alpha+u) = \cos\alpha+\frac{\sin^2\phi}{2M^2}\cos\alpha-\frac{\sin^4\phi}{8M^4}\cos 3\alpha+\dots.$$

Also we have

$$e^2 = \left[1-\frac{2\sin^2\phi\cos 2\alpha}{M^2}+\frac{\sin^4\phi}{M^4}\right]^{\frac{1}{2}}$$
$$= 1-\frac{\sin^2\phi\cos 2\alpha}{M^2}+\frac{\sin^4\phi\sin^2 2\alpha}{2M^4}+\frac{\sin^6\phi\sin 2\alpha\sin 4\alpha}{4M^6}+\dots.$$

Let us now consider how R_1 varies as ϕ increases from 0° to 90°. For brevity write $M\cos\phi=p$, and we then have

$$x_1 = \frac{2pc\cos(\alpha-u)}{p^2+c^2}$$

As M^2 is large, we have, *as a first approximation*, $c = 1$ and $u = 0$, so that

$$x_1 = \frac{2p \cos \alpha}{p^2+1} = \frac{2 \cos \alpha}{p+p^{-1}}.$$

This is a maximum when $p=1$, so that R_1 is then a minimum. Hence for light polarised perpendicularly to the plane of incidence the intensity of the reflected light diminishes as ϕ increases, until it reaches a minimum in the neighbourhood of $p=1$, which thus determines the "*quasi-polarising*" angle. When $p=1$ we have $x_1 = \cos \alpha$ (approximately) and $R_1=\sqrt{(1-x_1)/(1+x_1)} = \tan \frac{1}{2}\alpha$. For a certain class of steel we shall find later that $M^2=13$, and $\alpha=53° 42'$. In this case $p=1$, or $\cos \phi=M^{-1}$, gives $\phi=73° 54'$ as the quasi-polarising angle, and $R_1=0{\cdot}5062$ as the minimum value of the amplitude of the reflected wave. This approximation is, of course, somewhat rough, as we have neglected squares and higher powers of $1/M^2$. Proceeding to a higher order, we get

$$x_1 = \frac{2p\,(\cos \alpha - \frac{1}{2}M^{-2} \sin^2 \phi \cos 3\alpha - \frac{1}{8}M^{-4} \sin^4 \phi \cos 5\alpha)}{p^2+1-M^{-2} \sin^2 \phi \cos 2\alpha + \frac{1}{2}M^{-4} \sin^4 \phi \sin^2 2\alpha}.$$

In the small terms we may put the results of the first approximation $p=1$ and $\sin^2 \phi=1-M^{-2}$. This gives

$$x_1 = \frac{2p\,(\cos \alpha - \frac{1}{2}M^{-2} \cos 3\alpha + \frac{1}{2}M^{-4} \cos 3\alpha - \frac{1}{8}M^{-4} \cos 5\alpha)}{p^2+1-M^{-2} \cos 2\alpha + M^{-4} \cos 2\alpha + \frac{1}{2}M^{-4} \sin^2 2\alpha}.$$

This makes R_1 a minimum when $p^2 = 1-M^{-2} \cos 2\alpha + M^{-4} \cos 2\alpha + \frac{1}{2}M^{-4} \sin^2 2\alpha$. With the values of M and α given above for steel, this fixes the quasi-polarising angle at $\phi=73° 43'$.

In light polarised parallel to the plane of incidence we have

$$x_2 = \frac{2Mc \cos \phi \cos (\alpha+u)}{M^2c^2+\cos^2 \phi}.$$

As ϕ increases the denominator alters little, as $\cos^2 \phi$ is always small compared with M^2c^2, while the numerator steadily decreases. Thus R_2 increases steadily, and as the equation to determine its maxima or minima has no real roots, it has no maximum or minimum.

We have seen that the Principal Incidence is determined by the equation $\sin^4 \phi=M^2c^2 \cos^2 \phi$. Putting

$$c = 1-\tfrac{1}{2}M^{-2} \sin^2 \phi \cos 2\alpha + \tfrac{1}{8}M^{-4} \sin^4 \phi\,(2 \sin^2 2\alpha - \cos^2 2\alpha),$$

we get as the approximate equation to determine the Principal Incidence,

$$\sec \phi = M - \tfrac{1}{2}M^{-1} \sin^2 \phi \cos 2\alpha + \tfrac{1}{8}M^{-3} \sin^4 \phi\,(2 \sin^2 2\alpha - \cos^2 2\alpha) + M^{-1}(1+\tfrac{1}{2}M^{-2} \sin^2 \phi \cos 2\alpha).$$

This is most simply solved by approximations. The first approximation gives $\sec \phi=M$, so that $p=M \cos \phi=1$. To this order of approximation

the Principal Incidence and the quasi-polarising angle are the same, so that as a rule the Principal Incidence will be very near the quasi-polarising angle.

The second approximation gives

$$\sec\phi = M+M^{-1}(1-\tfrac{1}{2}\cos 2\alpha),$$

and the third,

$$\sec\phi = M+M^{-1}(1-\tfrac{1}{2}\cos 2\alpha)+M^{-3}\{\cos 2\alpha+\tfrac{1}{16}(1-3\cos 4\alpha)\}.$$

If $M = 4$, an error of $1/40$ in $\sec\phi$ corresponds to an error of about five minutes in ϕ, and this is within the limits of experimental error. In this case $1/M^3 = 1/64$, so that the second approximation will give the Principal Incidence sufficiently accurately for comparison with experimental results.

Returning now to the exact formulæ with which we began, we shall see how they fit in with modern experiments on metallic reflection. It has been usual* to compare them with the results of Jamin's experiments on steel and other metals. When this is done it is found that the theory fits in with the experiment as far as the main features of the metallic reflection are concerned; but it is only necessary to plot the experimental results to see that they are much too discordant to afford a satisfactory test.

Since Jamin's time there have been many experimental investigations into the phenomena of metallic reflection. Amongst others, we have an elaborate series of experiments by Sir John Conroy† on reflection from steel and speculum metal. Conroy found for steel that the Principal Incidence was 76° 20′ (with a probable error of ±5′) and the Principal Azimuth 28° 29′ (with a probable error of ±1′). He made four separate sets of experiments for reflection of light polarised perpendicularly and parallel to the plane of incidence. By rejecting the most discordant of the four when there is considerable discordance, and taking the mean of the remaining ones, and by taking the mean of the four where, as is generally the case, there is no great discordance, we get the following table, the notation being the same as at the outset of this paper:—

ϕ.	R_1.	R_2.	ϕ.	R_1.	R_2.
30°	0·7084	0·7791	70°	0·5152	0·9069
40	0·6803	0·8013	75	0·5047	0·9275
50	0·6401	0·8331	80	0·5254	0·9501
60	0·5837	0·8627			

* See, *e.g.*, Mascart's 'Traité d'Optique,' t. 2, vol. 13.

† See 'Roy. Soc. Proc.,' vol. 36, p. 187.

On plotting these numbers they will be found to be very fairly self-consistent, and to agree well with the observed value of the Principal Azimuth, the only exception being that R_1 appears to be too small at 80°.

There are two unknown constants, M and α, and these can be determined from a knowledge of the Principal Azimuth and Principal Incidence. At the Principal Incidence we have

$$Mc = \sin^2\phi/\cos\phi \quad \text{and} \quad \alpha+u = 2\beta.$$

Also

$$c^2 \sin 2u = M^{-2}\sin^2\phi \sin 2\alpha.$$

Thus

$$\sin 2u \tan^2\phi = \sin 2\alpha = \sin(4\beta-2u),$$

so that

$$\tan 2u = \frac{\sin 4\beta}{\tan^2\phi+\cos 4\beta}.$$

Taking $\phi = 76° \, 20'$ and $\beta = 28° \, 29'$, this gives $u = 1° \, 35'$ and $\alpha = 55° \, 23'$.

Also we have

$$\cot 2u = \frac{M^2-\sin^2\phi\cos 2\alpha}{\sin^2\phi \sin 2\alpha};$$

thus

$$M^2 = \frac{\sin^2\phi\sin 4\beta}{\sin 2u} = 15{\cdot}67.$$

This gives $\mu = M\cos\alpha = 2{\cdot}249$ and $a = M\sin\alpha = 3{\cdot}257$.

Having obtained α and M, we can calculate c and u for any value of ϕ from the formulæ

$$\cot 2u = \frac{M^2}{\sin^2\phi\sin 2\alpha} - \cot 2\alpha \quad \text{and} \quad c^2 = \frac{\sin^2\phi\sin 2\alpha}{M^2\sin 2u}.$$

Owing, however, to the smallness of u, the latter formula is not a very good one from which to determine c^2, since, particularly when ϕ is not large, the variations of cosec $2u$ are very rapid, so that a small error in u will affect c^2 considerably. We can avoid this difficulty by using the formula

$$c^4 = 1-2M^{-2}\sin^2\phi\cos 2\alpha+M^{-4}\sin^4\phi.$$

We thus get the following table:—

ϕ.	u.	c.	R_1 (theory).	Diff. from experiment.	R_2 (theory).	Diff. from experiment.
30°	0° 26′	1·003	0·7264	+0·0180	0·7879	+0·0088
40	0 42	1·005	0·6979	+0·0176	0·8106	+0·0093
50	0 59	1·006	0·6564	+0·0163	0·8389	+0·0058
60	1 16	1·009	0·5969	+0·0132	0·8728	+0·0101
70	1 29	1·012	0·5271	+0·0119	0·9111	+0·0042
75	1 34	1·012	0·5077	+0·0030	0·9322	+0·0047
80	1 37	1·014	0·5398	+0·0144	0·9541	+0·0040

We see from this table, and also from the graphical representation of these results in the figure below, that there is a considerable discrepancy between

FIG. 1.

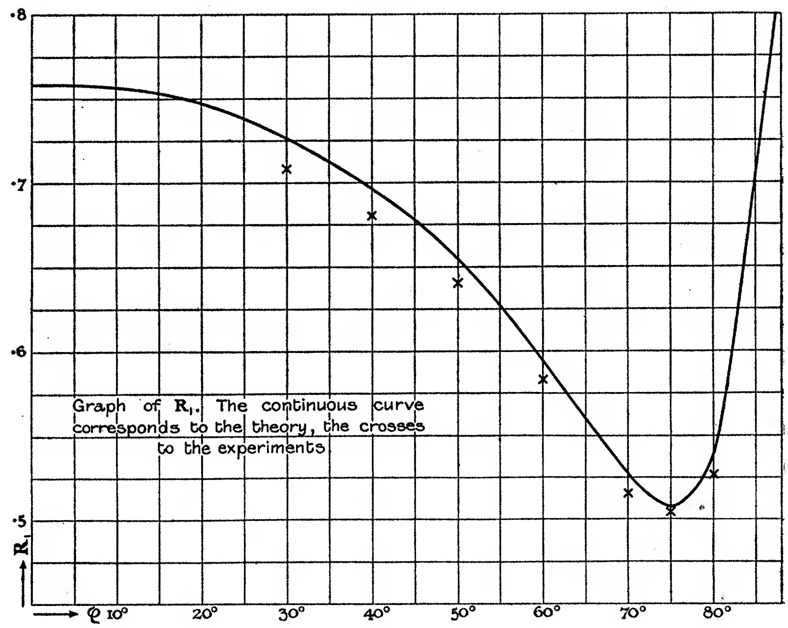
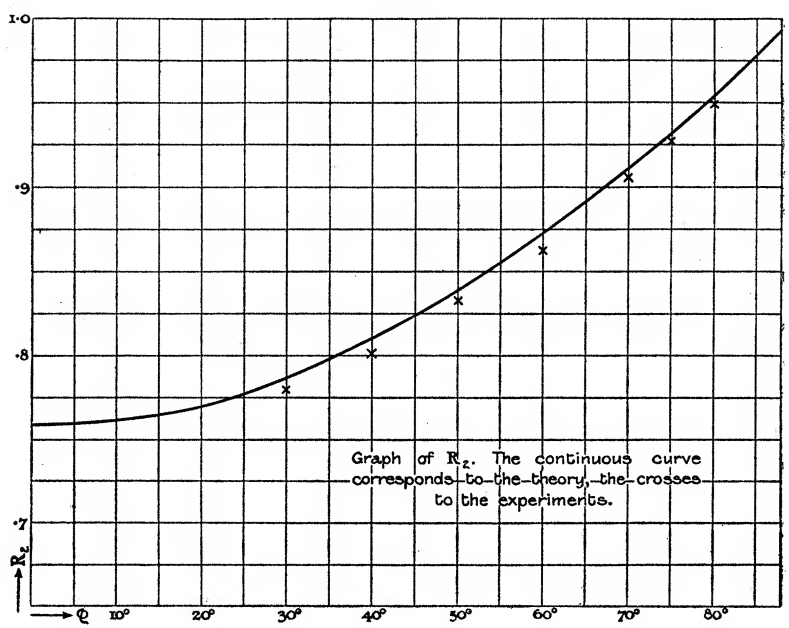


FIG. 2.



theory and experiment. The values of R_1 are almost uniformly between 1 and 2 per cent. larger than those found by experiment, while the values of R_2 are larger by quantities varying from between $\frac{1}{2}$ and 1 per cent. The differences are rather too large to be put down to experiment, and the fact that they are all of the same sign makes it improbable that they represent experimental errors. Moreover, on making a similar calculation for speculum metal, we find that the values are in this case always larger than those found by Conroy, the difference being greatest for R_2, where it amounts to $2\frac{1}{2}$ per cent. in some cases.

When dealing with transparent media it has been found that the discrepancies between theory and experiment have disappeared as soon as it has been recognised that the transition from one medium to another is, as a rule, gradual and not abrupt. In the present paper we shall investigate the extent to which this idea will assist us when dealing with the problem of metallic reflection.

When the layer of transition is taken into account, the formulæ for r_1 and r_2 take the forms*

$$r_1 = \frac{\kappa_0\mu_1^2/\mu_0^2 - \kappa_1 + id_1(\mathrm{F}\nu^2 - \mu_1^2 + \mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2)}{\kappa_0\mu_1^2/\mu_0^2 + \kappa_1 - id_1(\mathrm{F}\nu^2 - \mu_1^2 - \mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2)}$$

and
$$r_2 = \frac{\kappa_0 - \kappa_1 + id_1(\kappa_0\kappa_1 - \mathrm{E}\mu_1^2 + \nu^2)}{\kappa_0 + \kappa_1 + id_1(\kappa_0\kappa_1 + \mathrm{E}\mu_1^2 - \nu^2)}.$$

In our present notation $\nu = \sin\phi$, $\kappa_0 = \cos\phi$, $\kappa_1 = \mu\cos\phi'$, $\mu_0 = 1$, $\mu_1 = \mu$; so that we have

$$r_1 = \frac{\mu\cos\phi - \cos\phi' + id_1(\mu^{-1}\mathrm{F}\sin^2\phi - \mu + \mathrm{E}\mu^2\cos\phi\cos\phi')}{\mu\cos\phi + \cos\phi' - id_1(\mu^{-1}\mathrm{F}\sin^2\phi - \mu - \mathrm{E}\mu^2\cos\phi\cos\phi')}$$

and
$$r_2 = \frac{\cos\phi - \mu\cos\phi' + id_1(\mu\cos\phi\cos\phi' - \mathrm{E}\mu^2 + \sin^2\phi)}{\cos\phi + \mu\cos\phi' + id_1(\mu\cos\phi\cos\phi' + \mathrm{E}\mu^2 - \sin^2\phi)}.$$

Here $d_1 = 2\pi d/\lambda$, where d is the thickness of the layer, and λ the wave-length in air. E and F are complex constants defined by the equations $\mathrm{E} = \int_0^1 \mu^2 dx$ and $\mathrm{F} = \mu^2\int_0^1 \mu^{-2}dx$. The values of these constants depend, of course, on the law of variation of μ in the layer of transition. This being unknown, we cannot determine E and F, but we should expect them to lie between 1 and μ^2. If μ^2 had the value $\frac{1}{2}(1+\mu^2)$ in the layer, we should have $\mathrm{E} = \frac{1}{2}(1+\mu^2)$ and $\mathrm{F} = 2\mu^2/(1+\mu^2)$, so that if (as with steel) the modulus of μ^2 were about 13, $|\mathrm{E}|$ would be about 7 and $|\mathrm{F}|$ about 2. But in any case, owing to the largeness of μ^2, the term $\mathrm{E}\mu^2\cos\phi\cos\phi'$ in the formula for r_1 will be large compared with $\mu^{-1}\mathrm{F}\sin^2\phi - \mu$, except where ϕ is

* See a paper by the present writer, 'Roy. Soc. Proc.,' A, vol. 76, pp. 55 and 57.

very nearly 90° when the term introduced by the layer of transition becomes $\mu^{-1}F-\mu$, which is small compared with $E\mu^2$. The sequel will prove that d_1 is very small, so that the changes due to the layer will be small.

Putting $id_1 E\mu^2 = ae^{iw}$, neglecting the term $\mu^{-1}F\sin^2\phi-\mu$ in comparison with $E\mu^2\cos\phi\cos\phi'$, and putting $\cos\phi' = 1$ in the *small* terms, we get:—

$$r_1 = \frac{\mu\cos\phi-\cos\phi'+a\cos\phi\,.\,e^{iw}}{\mu\cos\phi+\cos\phi'+a\cos\phi\,.\,e^{iw}} = \frac{A_1e^{iv_1}+a\cos\phi\,.\,e^{iw}}{A_1'e^{iv_1'}+a\cos\phi\,.\,e^{iw}},$$

where $a\cos\phi$ is small compared with A_1 and A_1'.

If the modulus of r_1 be $R_1+\rho_1$, where R_1 is obtained from the formula of p. 212, then ρ_1 is the *correction* to R_1 due to the layer of transition.

We have

$$(R_1+\rho_1)^2 = \frac{A_1^2+a^2\cos^2\phi+2A_1\,a\cos\phi\cos(v_1-w)}{A_1'^2+a^2\cos^2\phi+2A_1'\,a\cos\phi\cos(v_1'-w)},$$

whence we have, approximately,

$$\rho_1 = R_1\,a\cos\phi\left[\frac{\cos(v_1-w)}{A_1}-\frac{\cos(v_1'-w)}{A_1'}\right].$$

Now
$$A_1e^{iv_1} = \mu\cos\phi-\cos\phi' = Me^{-i\alpha}\cos\phi-ce^{-iu},$$
$$A_1'e^{iv_1'} = \mu\cos\phi+\cos\phi' = Me^{-i\alpha}\cos\phi+ce^{-iu},$$

whence, employing the approximate values $c = 1$, $u = 0$, we have

$$A_1^2 = M^2\cos^2\phi+c^2-2M\cos\phi\cos\alpha,$$
$$A_1'^2 = M^2\cos^2\phi+1+2M\cos\phi\cos\alpha.$$

$$\tan v_1 = \frac{-M\cos\phi\sin\alpha}{M\cos\alpha\cos\phi-1}; \qquad \tan v_1' = -\frac{M\cos\phi\sin\alpha}{M\cos\alpha\cos\phi+1}.$$

$$\cos v_1 = \frac{1-M\cos\alpha\cos\phi}{A_1}; \qquad \cos v_1' = -\frac{1+M\cos\alpha\cos\phi}{A_1'},$$

$$\sin v_1 = \frac{M\cos\phi\sin\alpha}{A_1}; \qquad \sin v_1' = \frac{M\cos\phi\sin\alpha}{A_1'}.$$

Making these substitutions in the formula for ρ, we get

$$\rho_1 = R_1\,a\cos\phi\left[\cos w\left\{\frac{\cos v_1}{A_1}-\frac{\cos v_1'}{A_1'}\right\}+\sin w\left\{\frac{\sin v_1}{A_1}-\frac{\sin v_1}{A_1'}\right\}\right]$$
$$= R_1\,a\cos\phi\left[\cos w\left\{\frac{1-M\cos\alpha\cos\phi}{A_1^2}+\frac{1+M\cos\alpha\cos\phi}{A_1'^2}\right\}\right.$$
$$\left.+M\sin\alpha\cos\phi\sin w\left\{\frac{1}{A_1^2}-\frac{1}{A_1'^2}\right\}\right]$$
$$= \frac{2R_1\,a\cos\phi}{A_1^2A_1'^2}[\cos w-M^2\cos^2\phi\cos(2\alpha+w)].$$

This makes ρ_1 vanish when $\phi = 90°$ and also when

$$\cos\phi = M^{-1}\sqrt{\cos w/\cos(2\alpha+w)}.$$

In considering the law of variation of ρ, it is convenient to ascertain the position of its maxima and minima. Putting $M \cos\phi = p$ we have ρ_1 proportional to $R_1 p\,(p^2-b)/A_1^2 A_1'^2$, where $b = \cos w/\cos(2\alpha+w)$.

In the small terms we may use the approximate formulæ $c = 1$ and $u = 0$. We then have

$$A_1^2 A_1'^2 = (1+p^2)^2 - 4p^2\cos^2\alpha = 1-2p^2\cos 2\alpha + p^4,$$

and
$$R_1 = \sqrt{\frac{1-x_1}{1+x_1}}, \quad \text{where } x_1 = \frac{2p\cos\alpha}{1+p^2}.$$

Thus
$$-\frac{1}{R_1}\frac{dR_1}{dp} = \frac{1}{1-x_1^2}\frac{dx_1}{dp} = \frac{2\cos\alpha\,(1-p^2)}{1-2p^2\cos 2\alpha+p^4}.$$

The equation to determine the maxima and minima is, then,

$$\frac{3p^2-b}{p\,(q^2-b)} = \frac{2\cos\alpha\,(1-p^2)+4p\,(p^2-\cos 2\alpha)}{1-2p^2\cos 2\alpha+p^4},$$

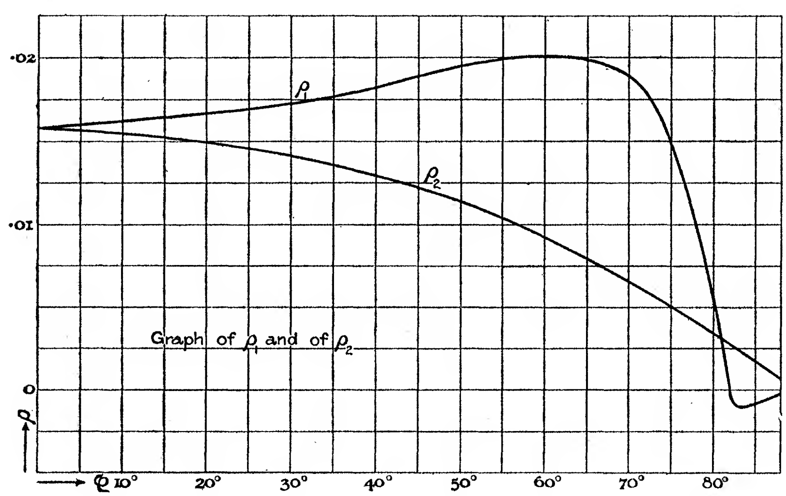
i.e., $$p^6 - 2\cos\alpha\,.\,p^5 + (2\cos 2\alpha - 3b)\,p^4 + 2\,(1+b)\cos\alpha\,.\,p^3 + (2b\cos 2\alpha - 3)\,p^2 - 2b\cos\alpha\,.\,p + b = 0.$$

With the values of α and w found later for steel this equation becomes

$$p^6 - 1{\cdot}176p^5 - 0{\cdot}8178p^4 + 1{\cdot}254p^3 - 3{\cdot}041p^2 - 0{\cdot}0783p + 0{\cdot}0666 = 0.$$

Solving this, by Horner's process, we get two real roots, $p = 0{\cdot}37$ and $p = 1{\cdot}8$. The former corresponds to $\phi = 84°\ 6'$ and determines the position of the minimum, the latter corresponds to $\phi = 60°$ and determines the position of the maximum. The graph below (fig. 3) represents the march of the function ρ_1 in the case of steel.

Fig. 3.



In dealing similarly with r_2 we note that the modulus of $E\mu^2$ is large compared with that of $\mu\cos\phi\cos\phi'+\sin^2\phi$. Neglecting the latter term in comparison with the former we get

$$r_2 = \frac{\cos\phi-\mu\cos\phi'-id_1E\mu^2}{\cos\phi+\mu\cos\phi'+id_1E\mu^2} = \frac{A_2e^{iv_2}-ae^{iw}}{A_2'e^{iv_2'}+ae^{iw}}.$$

If the modulus of r_2 be $R_2+\rho_2$, so that ρ_2 is the *correction* due to the layer of transition, we have

$$\rho_2 = -R_2a\left[\frac{\cos(v_2-w)}{A_2}+\frac{\cos(v_2'-w)}{A_2'}\right],$$

$$A_2e^{iv_2} = \cos\phi-\mu\cos\phi'; \qquad A_2'e^{iv_2'} = \cos\phi+\mu\cos\phi',$$

whence, approximately, $A_2^2 = M^2+\cos^2\phi-2M\cos\phi\cos\alpha,$

$$A_2'^2 = M^2+\cos^2\phi+2M\cos\phi\cos\alpha;$$

$$\tan v_2 = \frac{M\sin\alpha}{\cos\phi-M\cos\alpha}; \qquad \tan v_2' = \frac{-M\sin\alpha}{\cos\phi+M\cos\alpha};$$

so that
$$\rho_2 = \frac{2R_2a\cos\phi}{A_2A_2'^2}[\cos^2\phi\cos w-M^2\cos(2\alpha+w)],$$

and ρ_2 will thus vanish when $\phi = 90°$. It would also vanish if $\cos^2\phi = M^2\cos(2\alpha+w)/\cos w$; but, as a rule, this will be larger than unity, and so there will be no real value of ϕ to satisfy it.

In the formula for ρ_2 the term $\cos^2\phi\cos w$ will usually be negligible in comparison with $M^2\cos(2\alpha+w)$, so that we shall have, approximately,

$$\rho_2 = -2R_2\,a\cos(2\alpha+w)\cos\phi/A_2^2A_2'^2.$$

Now
$$A_2^2A_2'^2 = (M^2+\cos^2\phi)^2-4M^2\cos^2\phi\cos^2\alpha$$
$$= \cos^4\phi+M^4+2M^2\cos^2\phi\,(1-2\cos^2\alpha).$$

For almost all the metals α is greater than 45°, so that $1-2\cos^2\alpha$ is positive. Thus $A_2^2A_2'^2$ is greater than M^4 and ρ_2 is less than

$$2M^{-2}R_2\,a\cos\phi\cos(2\alpha+w).$$

Owing to the factor M^{-2}, ρ_2 will thus be small and it will diminish steadily with ϕ. If we investigate the position of the maxima and minima in the same manner as we did with ρ_1 we are led to the following equation :—

$$\left(\frac{4}{M^6}-\frac{3}{M^8}\right)p^6-\frac{2\cos\alpha}{M^6}p^5+\left(\frac{6\cos2\alpha}{M^4}-\frac{4b}{M^6}+\frac{b}{M^8}\right)p^4$$
$$+\left(\frac{2\cos\alpha}{M^2}-\frac{4\cos2\alpha}{M^4}+\frac{2b\cos\alpha}{M^6}\right)p^3-\left(3-\frac{2b\cos2\alpha}{M^4}\right)p^2-\frac{2b\cos\alpha}{M^2}p-6 = 0.$$

As $p = M\cos\phi$, p cannot be greater than M, and the above equation in p has no real roots less than M, so that there are no maxima and minima.

The graph above (fig. 3) represents the march of the function ρ_2 in the case of steel. It will be seen from this figure that the range for which ρ_1 and ρ_2 are appreciable is much larger than in the case of a transparent medium, where the influence of the transition layer is practically confined within a few degrees of the polarising angle.

It has been observed before that for many purposes the *ratio* of R_1 to R_2, and the *difference* of phase between light polarised perpendicularly and parallel to the plane of incidence is what is wanted for comparison with experiment. We proceed to develop some formulæ suitable for this end.

We have

$$r_1 = \frac{\mu\cos\phi - \cos\phi' + id_1 E\mu^2\cos\phi\cos\phi'}{\mu\cos\phi + \cos\phi' + id_1 E\mu^2\cos\phi\cos\phi'}$$

$$= \frac{\sin(\phi-\phi')\cos(\phi+\phi') + a\sin\phi'\cos\phi'\cos\phi e^{iw}}{\sin(\phi+\phi')\cos(\phi-\phi') + a\sin\phi'\cos\phi'\cos\phi e^{iw}};$$

$$r_2 = \frac{\cos\phi - \mu\cos\phi' - id_1 E\mu^2}{\cos\phi + \mu\sin\phi' + id_1 E\mu^2} = -\frac{\sin(\phi-\phi') + a\sin\phi' e^{iw}}{\sin(\phi+\phi') + a\sin\phi' e^{iw}}.$$

Thus

$$\frac{r_1}{r_2} = -\frac{\sin(\phi-\phi')\cos(\phi+\phi') + a\sin\phi'\cos\phi'\cos\phi e^{iw}}{\sin(\phi+\phi')\cos(\phi-\phi') + a\sin\phi'\cos\phi'\cos\phi e^{iw}} \times \frac{\sin(\phi+\phi') + a\sin\phi' e^{iw}}{\sin(\phi-\phi') + a\sin\phi' e^{iw}}$$

$$= -\frac{A+a'}{A'+a'}\cdot\frac{B+b'}{B'+b'},$$ where the moduli of a' and b' are small compared with those of A and B,

$$= -\frac{AB}{A'B'}\left[1 + a'\left(\frac{1}{A} - \frac{1}{A'}\right) + b'\left(\frac{1}{B} - \frac{1}{B'}\right)\right]$$

$$= -\frac{\cos(\phi+\phi')}{\cos(\phi-\phi')}\left[1 + a\sin\phi'\cos\phi'\cos\phi e^{iw}\left\{\frac{1}{\sin(\phi-\phi')\cos(\phi+\phi')} - \frac{1}{\sin(\phi+\phi')\cos(\phi-\phi')}\right\} + a\sin\phi' e^{iw}\left\{\frac{1}{\sin(\phi+\phi')} - \frac{1}{\sin(\phi-\phi')}\right\}\right]$$

$$= -\frac{\cos(\phi+\phi')}{\cos(\phi-\phi')}\left[1 + \frac{2a\sin^2\phi'\cos\phi\,.\,e^{iw}\{\cos^2\phi' - \cos(\phi-\phi')\cos(\phi+\phi')\}}{\sin(\phi+\phi')\cos(\phi+\phi')\sin(\phi-\phi')\cos(\phi-\phi')}\right]$$

$$= -\frac{\cos(\phi+\phi')}{\cos(\phi-\phi')}\left[1 - \frac{2a\sin^2\phi'\sin^2\phi\cos\phi\, e^{iw}}{\sin(\phi+\phi')\cos(\phi+\phi')\sin(\phi-\phi')\cos(\phi-\phi')}\right]$$

$$= R_1/R_2\,.\,e^{i(\theta_1-\theta_2)}[1 - qe^{i\theta}],$$

where $$qe^{i\theta} = \frac{2a\sin^2\phi'\sin^2\phi\cos\phi e^{iw}}{\sin(\phi+\phi')\cos(\phi+\phi')\sin(\phi-\phi')\cos(\phi-\phi')}.$$

Now

$$\sin(\phi+\phi')\cos(\phi+\phi')\sin(\phi-\phi')\cos(\phi-\phi') = \tfrac{1}{2}(\sin 2\phi + \sin 2\phi') \times \tfrac{1}{2}(\sin 2\phi - \sin 2\phi')$$

$$= \tfrac{1}{4}(\sin^2 2\phi - \sin^2 2\phi') = \sin^2\phi\cos^2\phi - \sin^2\phi'\cos^2\phi',$$

and remembering that

$$\cos\phi' = ce^{-iu} \quad \text{and} \quad \sin\phi' = \mathrm{M}^{-1}e^{ia}\sin\phi,$$

we get

$$qe^{i\theta} = \frac{2a\sin^2\phi\cos\phi\,.\,e^{i(2\alpha+w)}}{\mathrm{M}^2\cos^2\phi - c^2e^{2i(\alpha-u)}}.$$

Hence

$$q = \frac{2a\sin^2\phi\cos\phi}{[\mathrm{M}^4\cos^4\phi + c^4 - 2\mathrm{M}^2c^2\cos^2\phi\cos 2(\alpha-u)]^{\frac{1}{2}}}$$

$$= 2a\sin^2\phi\cos\phi/\mathrm{A}_1\mathrm{A}_1', \text{ in the notation of p. 219,}$$

$$\theta = 2\alpha + w + \theta', \quad \text{where} \quad \tan\theta' = \frac{c^2\sin 2(\alpha-u)}{\mathrm{M}^2\cos^2\phi - c^2\cos 2(\alpha-u)}.$$

To determine the maximum value of q we put $\mathrm{M}\cos\phi = p$, as before, and using the approximate relations $c = 1$ and $u = 0$, we have to make $p(1-p^2/\mathrm{M}^2)/\sqrt{p^4-2p^2\cos 2\alpha+1}$ a maximum.

This requires

$$(p^4-2p^2\cos 2\alpha+1)(1-3p^2\mathrm{M}^{-2}) - 2p^2(1-p^2\mathrm{M}^{-2})(p^2-\cos 2\alpha) = 0.$$

Solving this by approximations we get $p = 1$ as the first approximation. This, as we have seen (p. 214), is the first approximation to the quasi-polarising angle. A second approximation gives $p^4 = 1 - 8\mathrm{M}^{-2}\sin^2\alpha$. If $\mathrm{M}^2 = 13$ and $\alpha = 53° \, 42'$, as we shall find later for a certain class of steel, the first approximation gives $\phi = 73° \, 54'$, and the second $\phi = 75° \, 52'$ which is very near the Principal Incidence.

If we put $\quad 1 - qe^{i\theta} = se^{i\chi}, \quad$ we have $\quad \dfrac{\mathrm{R}_1+\rho_1}{\mathrm{R}_2+\rho_2} = s\dfrac{\mathrm{R}_1}{\mathrm{R}_2},$

and the difference of phase is $\theta_1 - \theta_2 + \chi$.

Here $\quad s = \sqrt{1-2q\cos\theta+q^2} \quad$ and $\quad \tan\chi = \dfrac{-q\sin\theta}{1-q\cos\theta}.$

If, as will usually be the case, q be small, χ will be small and the correction to the change of phase will be small. It will be greatest when q is largest, *i.e.*, in the neighbourhood of the Principal Incidence.

Perhaps, however, the most important point to notice is that, even although the correction to the change of phase be small, it may make an appreciable difference to the position of the Principal Incidence. At the Principal Incidence we have

$$\theta_1 - \theta_2 + \chi = \tfrac{1}{2}\pi, \quad \text{therefore} \quad \cot(\theta_1-\theta_2) = \tan\chi.$$

or

$$\sin^4\phi - \mathrm{M}^2c^2\cos^2\phi = 2\mathrm{M}c\cos\phi\sin^2\phi\sin(\alpha+u)\tan\chi.$$

This is the equation to determine the Principal Incidence, and owing to the presence of M on the right hand side, that term may be appreciable even although χ be small.

It would seem then, that two constants (M and α) are not sufficient to describe the optical properties of a metal—at least two more (a and w) are required in the problem of reflection, these latter constants depending on the law of variation of μ^2 in the layer of transition between the media. This being the case, we cannot derive the optical constants from observations of the Principal Azimuth and Principal Incidence alone. The simplest method is to proceed by successive approximations. The true values of M and α will be smaller than those obtained by neglecting the layer of transition and proceeding as on p. 216. On obtaining approximate values of M and α in this way, we can calculate the constants a and w from observation of the Principal Azimuth (β) and Principal Incidence (ϕ).

We have

$$\sqrt{1-2q\cos\theta+q^2} = \frac{R_1+\rho_1}{R_2+\rho_2}\Big/\frac{R_1}{R_2} = \frac{R_2}{R_1}\tan\beta, \tag{i}$$

and

$$\frac{-q\sin\theta}{1-q\cos\theta} = \tan\chi = \frac{\sin^4\phi - Mc^2\cos^2\phi}{2\,Mc\cos\phi\sin^2\phi\sin^2(\alpha+u)} \tag{ii}$$

These are two equations, from which q and θ may be readily determined.

Equation (ii) determines χ, and we then have

$$\tan\theta = \frac{R_2/R_1 \,.\, \tan\beta\sin\chi}{R_2/R_1 \,.\, \tan\beta\cos\chi - 1}; \qquad q = -R_2/R_1 \,.\, \tan\beta\sin\chi\operatorname{cosec}\theta,$$

and once q is determined we derive a from the equation

$$q = 2a\sin^2\phi\cos\phi/A_1A_1'.$$

We shall apply this method to Conroy's experiments on reflection from steel. The values of M^2 and α already obtained (p. 216) are too high. We shall take as the next approximation $M^2 = 14{\cdot}44$ and $\alpha = 54°$, although the sequel will prove that these are still too high. Taking $\beta = 28° \; 29'$ and $\phi = 76° \; 20'$ from Conroy's experiments, we get the following from the equations just obtained :—

$$\chi = 2° \; 42'; \qquad \theta = 180° + 64° \; 32'$$
$$q = 0{\cdot}0535; \qquad a = 0{\cdot}1742;$$
$$\theta' = 42° \; 59'; \qquad w = 93° \; 33'.$$

With $M^2 = 14{\cdot}44$ and $\alpha = 54°$ the formulæ of p. 212 give us the following table :—

ϕ.	u.	c.	R_1.	R_2.
0°	0° 0′	1	0·7423	0·7423
30	0 28	1·007	0·7074	0·7745
40	0 46	1·007	0·6790	0·7979
50	1 6	1·007	0·6352	0·8279
60	1 23	1·009	0·5751	0·8636
70	1 38	1·010	0·5063	0·9046
75	1 43	1·010	0·4911	0·9268
80	1 47	1·011	0·5304	0·9504

The formulæ for ρ_1 and ρ_2 on pp. 219 and 221, with the above values of a, R_1, and R_2, give us the the following:—

ϕ.	ρ_1.	ρ_2.	ϕ.	ρ_1.	ρ_2.
0°	0·0158	0·0158	60°	0·0206	0·0092
30	0·0172	0·0141	70	0·0191	0·0066
40	0·0182	0·0129	75	0·0150	0·0051
50	0·0195	0·0114	80	0·0060	0·0035

These results are represented graphically in fig. 3 above.

If we take these values of R_1, R_2, ρ_1, and ρ_2, and compare $R_1 + \rho_1$ and $R_2 + \rho_2$ with the results of Conroy's experiments as set out on p. 215, we shall find that R_1 and R_2 are still too large to fit in with the experimental data. This indicates that we must further depress M and α; but now that the correction ρ_1 and ρ_2 have been approximately estimated, there is not the same guesswork in seeking for the correct values of M and α. As a second approximation we shall take $M^2 = 13$ and $\alpha = 53° 42'$. With these we get:—

ϕ.	R_1.	R_2.	ϕ.	R_1.	R_2.
0°	0·7300	0·7300	60°	0·5611	0·8557
30	0·6942	0·7630	70	0·4978	0·8991
40	0·6639	0·7874	75	0·4893	0·9230
50	0·6204	0·8183	80	0·5368	0·9473

The change of M and α will, of course, affect all the other quantities including the corrections ρ_1 and ρ_2. The alterations will not amount, however, to more than 1 per cent., and such a fraction of ρ_1 and ρ_2 is scarcely

worth considering in view of the uncertainty of the experimental results. We shall, therefore, take ρ_1 and ρ_2 to have the same values as those calculated for $M^2 = 14{\cdot}44$ and $\alpha = 54°$. This will give us the following table, in which the results are compared with Conroy's numbers set out on p. 215.

ϕ.	$R_1+\rho_1$. Theory.	$R_1+\rho_1$. Experiment.	Difference.
30°	0·7114	0·7084	+0·0030
40	0·6821	0·6803	+0·0018
50	0·6399	0·6401	−0·0002
60	0·5817	0·5837	−0·0020
70	0·5169	0·5152	+0·0017
75	0·5043	0·5047	−0·0004
80	0·5428	0·5254	+0·0174

ϕ.	$R_2+\rho_2$. Theory.	$R_2+\rho_2$. Experiment.	Difference.
30°	0·7771	0·7791	−0·0020
40	0·8003	0·8013	−0·0010
50	0·8297	0·8331	−0·0034
60	0·8649	0·8627	+0·0022
70	0·9057	0·9069	−0·0012
75	0·9281	0·9275	+0·0006
80	0·9508	0·9501	+0·0007

It will be seen that the differences between theory and experiment are well within the limits of experimental error. The only appreciable difference is for R_1 at 80°, and we have already noted (p. 218) that there is reason to doubt the accuracy of the experiments in this case. The results are exhibited graphically in figs. 4 and 5.

The difference of phase between the light polarised parallel, and that polarised perpendicularly to the plane of incidence is $\theta_2 - \theta_1 - \chi$, where

$$\tan(\theta_2 - \theta_1) = \frac{2Mc\cos\phi\sin^2\phi\sin(\alpha+u)}{M^2c^2\cos^2\phi - \sin^4\phi};$$

$$\tan\chi = \frac{-q\sin\theta}{1-q\cos\theta}; \qquad q = \frac{2a\sin^2\phi\cos\phi}{A_1A_1'};$$

$$\theta = 2\alpha + w + \theta'; \qquad \tan\theta' = \frac{c^2\sin 2(\alpha-u)}{M^2\cos^2\phi - c^2\cos 2(\alpha-u)}.$$

Fig. 4.

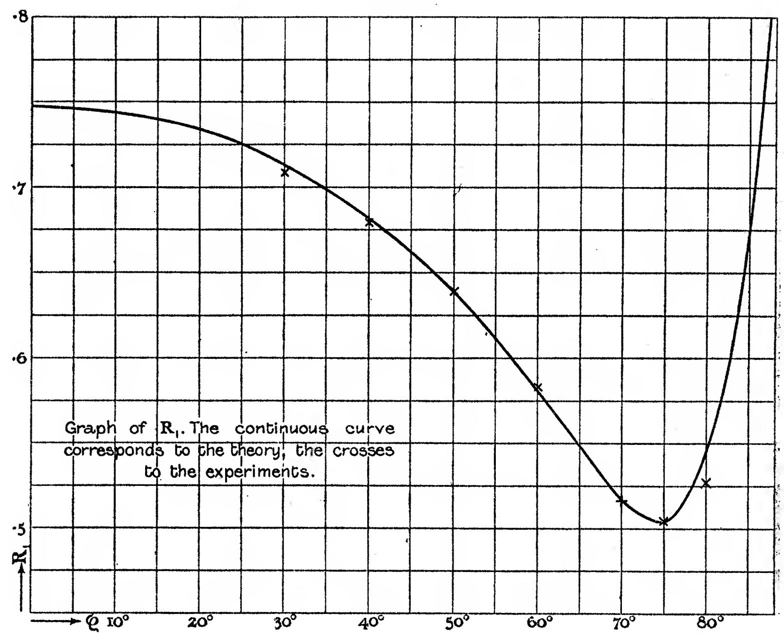
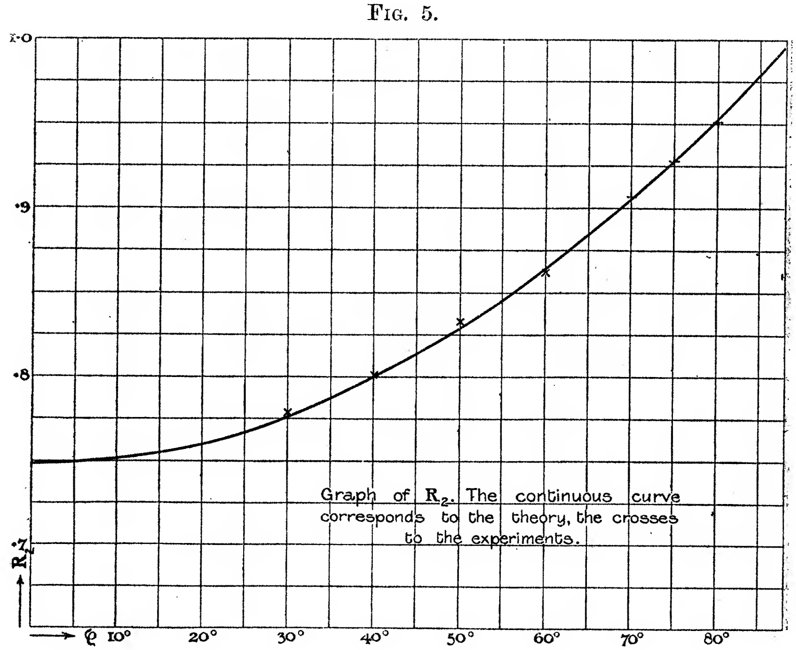

·8
·7
·6
·5
R₁
φ 10° 20° 30° 40° 50° 60° 70° 80°
Graph of R₁. The continuous curve corresponds to the theory, the crosses to the experiments.

Fig. 5.


1·0
·9
·8
·7
R₂
φ 10° 20° 30° 40° 50° 60° 70° 80°
Graph of R₂. The continuous curve corresponds to the theory, the crosses to the experiments.

From these, with the numerical values of the constants adopted above, we derive the following:—

ϕ.	θ'.	$\theta'-180°$.	ϕ.	θ'.	$\theta'-180°$.
30°	5° 32′	26° 29′	70°	29° 13′	50° 10′
40	7 2	27 59	75	41 35	62 32
50	9 51	30 48	80	57 22	78 19
60	15 40	36 37			

While for q and χ we get:—

ϕ.	q.	χ.	ϕ.	q.	χ.
0°	0	0° 0′	60°	0·0327	1° 5′
30	0·0068	0 10	70	0·0482	2 3
40	0·0125	0 20	75	0·0535	2 39
50	0·0208	0 36	80	0·0486	2 42

From these we derive the following table for the difference of phase. The column headed $\delta\,(=\theta_2-\theta_1)$ gives the *uncorrected* difference of phase in degrees, while that headed $\delta-\chi$ gives the *corrected* difference. The columns headed δ/π and $(\delta-\chi)/\pi$ give the corresponding phase differences as fractions of the half wave-length.

ϕ.	δ.	$\delta-\chi$.	δ/π.	$(\delta-\chi)/\pi$.
0°	0° 0′	0° 0′	0	0
30	7 24	7 14	0·0411	0·0402
40	13 54	13 34	0·0772	0·0754
50	23 42	23 6	0·1317	0·1283
60	39 11	38 6	0·2177	0·2116
70	66 54	64 51	0·3718	0·3604
75	89 16	86 37	0·4960	0·4814
80	118 6	115 24	0·6561	0·6410
90	180 0	180 0	1	1

Unfortunately, Conroy did not observe the differences of phase, so that we are unable to put these numbers to the test of agreement with experiment. From our formulæ we see that the phase differences will vary with M and α, and so will depend on the optical quality of the steel and on the nature of the light employed.* For want of other data we shall compare our formulæ with the experimental results obtained by M. de Senarmont, taking

* *Cf.* the results of a long series of experiments by M. Mouton, quoted in Mascart's 'Traité d'Optique,' t. 2, p. 542.

$M^2 = 15{\cdot}5$ and $\alpha = 54°$. With these constants we get the following values for δ:—

ϕ	30°.	40°.	50°.	60°.	70°.	75°.	80°.
δ	6° 48′	12° 29′	21° 44′	35° 55′	61° 38′	83° 10′	119°

The Principal Incidence was very close to that obtained by Conroy, and the values of M and α are near those that we have found to correspond closely with the results of Conroy's experiments. We might expect, then, that the corrections due to the layer of transition would not differ much from those found for Conroy's steel. Taking them to be the same, we get the following table, in which the theoretical results are compared with the experimental, and the differences noted.

ϕ.	$(\delta-\chi)$.		Difference.	$(\delta-\chi)/\pi$.		Difference.
	Theory.	Experiment.		Theory.	Experiment.	
30°	6° 38′	6° 37′	+0° 1′	0·0369	0·0368	+0·0001
40	12 9	12 0	+0 9	0·0675	0·0667	+0·0008
50	21 8	20 38	+0 30	0·1174	0·1147	+0·0027
60	34 50	32 6	+2 44	0·1936	0·1783	+0·0153
70	59 35	56 59	+2 36	0·3311	0·3165	+0·0145
75	80 31	80 46	−0 15	0·4472	0·4489	−0·0017
80	116 18	116 42	−0 24	0·6463	0·6485	−0·0022

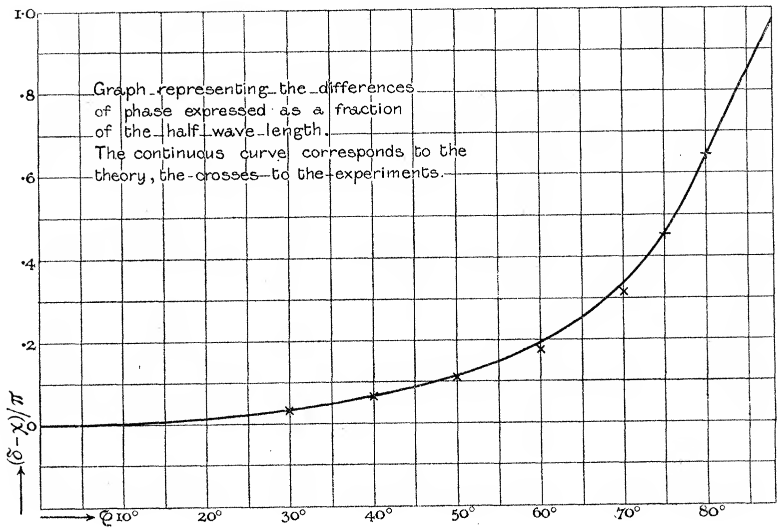
These results are exhibited graphically in fig. 6 (p. 230), and it will be seen that there is a satisfactory agreement between theory and experiment.

Thickness of the Transition Layer.

If d be the thickness of the layer, we have $d_1 = 2\pi d/\lambda_0\mu_0$, where μ_0 is the coefficient of refraction of the medium in contact with the metal, and λ_0 the wave-length in that medium.

Also we have $ae^{iw} = id_1\mathrm{E}\mu^2$, so that $a = d_1\mathrm{E}_1\mathrm{M}^2$, where E_1 is the modulus of E. From these relations we see that in order to determine the thickness of the layer we must know E_1 as well as a and M^2. As the value of E_1 depends on the law of variation of μ^2, we cannot determine it without a knowledge of the physical condition of the layer. We have seen, however (p. 218), that we should expect E_1 to lie between 1 and M^2. This would make d_1 lie between a/M^2 and a/M^4.

FIG. 6.



With the constants adopted above in the case of steel when discussing Conroy's experiments, this makes d_1 lie between 0·0132 and 0·001. The upper limit is near that derived from Kurz's experiments on reflection from glass into air.* It gives $d/\lambda = 0{\cdot}0021$.

If μ^2 in the layer obeyed any simple law there would be no difficulty in calculating E. Suppose, for example, that $\mu^2 = [\mu_0{}^2 + (\mathrm{M}^2 - \mu_0{}^2)\,x]\,e^{-i2\alpha x}$ which gives the correct values of μ^2 at the faces of the layer. We should then have

$$\mathrm{E}_1 e^{i\omega} = \mathrm{E} = \int_0^1 \mu^2 dx = \int_0^1 [\mu_0{}^2 + (\mathrm{M}^2 - \mu_0{}^2)\,x]\,e^{-i2\alpha x} dx$$

$$= \left[\frac{\mathrm{M}^2 \sin 2\alpha}{2\alpha} - \frac{\mathrm{M}^2 - \mu_0{}^2}{2}\left(\frac{\sin\alpha}{\alpha}\right)^2\right] + i\left[\frac{\mathrm{M}^2 - \mu_0{}^2}{2}\left(\frac{1 - \sin 2\alpha/2\alpha}{\alpha}\right) - \mathrm{M}^2 \frac{\sin^2\alpha}{\alpha}\right].$$

Whence

$$\mathrm{E}_1{}^2 = \left[\frac{\mathrm{M}^2 \sin 2\alpha}{2\alpha} - \frac{\mathrm{M}^2 - \mu_0{}^2}{2}\left(\frac{\sin\alpha}{\alpha}\right)\right]^2 + \left[\frac{\mathrm{M}^2 - \mu_0{}^2}{2}\left(\frac{1 - \sin 2\alpha/2\alpha}{\alpha}\right) - \frac{\mathrm{M}^2 \sin^2\alpha}{\alpha}\right]^2,$$

and
$$\tan\omega = \frac{(\mathrm{M}^2 - \mu_0{}^2)(1 - \sin 2\alpha/2\alpha) - 2\mathrm{M}^2 \sin^2\alpha}{\mathrm{M}^2 \sin 2\alpha - (\mathrm{M}^2 - \mu_0{}^2)\sin^2\alpha/\alpha}.$$

In Conroy's steel we have

$$\mathrm{M}^2 = 13\,; \quad \alpha = 0{\cdot}9372 \text{ radian } (= 53^\circ\ 42').$$

This gives for air,

$$\mu_0 = 1, \qquad \mathrm{E}_1 = 6{\cdot}261, \quad \omega = 110^\circ\ 24'$$

* See 'Roy. Soc. Proc.,' A, vol. 76, p. 58.

and for water,

$$\mu_0 = 1{\cdot}33, \quad E_1 = 6{\cdot}552, \quad \omega = 112^\circ \; 5'.$$

In the case of air we get $w = 90^\circ + \omega - 2\alpha = 93^\circ$, which is within half a degree of the value derived (on p. 224) from Conroy's experiments. The corresponding value of E_1 makes $d_1 = 0{\cdot}0021$ and $d/\lambda = 0{\cdot}0003$.

Effect of Changing the Medium in Contact with the Metal.

It has been found by experiment that the Principal Incidence and Principal Azimuth depend not only upon the nature of the reflecting metal, but also upon the medium in contact with it. Quincke* made some investigations into this matter, but the subject has been discussed much more completely in a long series of experiments on gold and silver by Sir John Conroy.† Conroy could find no simple relation between the changes in the values of the Principal Azimuth and Incidence and the indices of the media. His results, however, are in complete accordance with the trend of this paper, as is also the observation of Drude that surface impurities tend to reduce the value of the Principal Incidence.‡

If the medium in contact with the metal be of refractive index μ_0, we have to replace M in our earlier formulæ by M/μ_0, keeping α as before. We have seen that for an *abrupt* transition the Principal Incidence is given by the formula $\sin^4\phi = M^2c^2\cos^2\phi$, which is approximately equivalent§ to $\sec\phi = M + M^{-1}(1 - \frac{1}{2}\cos 2\alpha)$. If there be a layer of transition we have the equation

$$\sin^4\phi = M^2c^2\cos^2\phi + 2Mc\cos\phi\sin^2\phi\sin(\alpha+u)\tan\chi.$$

Now $\tan\chi$ is always small, so that in the last term we may put $\sin^2\phi = Mc\cos\phi$ and we then get

$$\sin^4\phi = M^2c^2\cos^2\phi\,(1+2\kappa),$$

where $\kappa = \sin(\alpha+u)\tan\chi$, and is small.

Thus the effect of the layer is to replace M^2 by $M^2(1+2\kappa)$ or M by $M(1+\kappa)$, while the effect of replacing air by a medium of refractive index μ_0 is to replace M by M/μ_0.

Hence, if there be a layer of transition between a medium μ_0 and the metal, the Principal Incidence will be determined by the equation

$$\sec\phi = M' + M'^{-1}(1 - \tfrac{1}{2}\cos 2\alpha), \quad \text{where} \quad M' = M(1+\kappa)\,\mu_0^{-1}$$

* 'Pogg. Ann.,' vol. 188, p. 541.

† See 'Roy. Soc. Proc.,' vol. 31, p. 486.

‡ See Drude, 'Wied. Ann.,' vol. 36, 1889, and vol. 39, 1890.

§ See p. 213.

An increase of μ_0 will diminish M′ and so diminish ϕ, the Principal Incidence.

The following are the means of Conroy's experimental determinations of the Principal Incidences, the incident light being yellow:—

	°	′		°	′
Gold in air	71	43	Silver in air	74	37
„ water	67	39	„ water	72	15
„ carbon bisulphide	66	36	„ carbon tetrachloride	71	39

A change in μ_0 will also affect the Principal Azimuth (β). We have

$$\tan\beta = (1-2q\cos\theta+q^2)^{\frac{1}{2}}\tan\tfrac{1}{2}(\alpha+u)$$

$$= (1-q\cos\theta)\tan\tfrac{1}{2}(\alpha+u), \text{ very nearly.}$$

The angle u is determined by the equation

$$\cot 2u = \frac{M^2/\mu_0^2}{\sin^2\phi\sin 2\alpha} - \cot 2\alpha.$$

An increase of μ_0 will diminish $\cot 2u$ and therefore increase u, so that as a rule β will be increased slightly, although in some cases the increase of $\tan\frac{1}{2}(\alpha+u)$ may be counterbalanced by the diminution of the factor $1-q\cos\theta$.

These are the means of Conroy's experimental determinations of the Principal Azimuth, corresponding to the Principal Incidences above:—

	°	′		°	′
Gold in air	41	14	Silver in air	43	22
„ water	41	15	„ water	44	9
„ carbon bisulphide	41	41	„ carbon tetrachloride	43	40

It will be observed that these results are in general accord with the argument above. Unfortunately, however, we have not sufficient data to put the theory of a layer of transition to the exact test of numerical verification or otherwise. We have seen that a knowledge of the Principal Incidence and the Principal Azimuth is not enough to determine the optical properties of a metal, and Conroy's results do not enable us to supply the deficiency. Even if we had sufficient data to determine the constants a and w for air, we could still do little better than guess what they would become when some other medium was in contact with the metal. We have $E = \int_0^1 \mu^2 dx$, so that the modulus of E should be increased by an increase of μ_0.*

* *Cf.* p. 230.

Hence an increase of μ_0 might be expected to raise a and therefore also χ and κ. These quantities χ and κ will be further raised when M is replaced by M/μ_0, for this will increase θ', and so increase $\sin\theta$ and diminish $-\cos\theta$. We should expect, then, that the diminution of ϕ, due to the increase of μ_0, would be greater if there were a layer of transition than if there were none. The effect on the Principal Azimuth (β) is not so easily described. An increase of μ_0 will raise q, but it will diminish $\cos\theta$, so that we cannot say in general whether $1-q\cos\theta$ will be increased or diminished.

A little investigation will show that Conroy's experimental results are not consistent with the theory of an abrupt transition from one medium to the other. Thus for gold in air with yellow light his values of the Principal Azimuth and Incidence would give $M = 2{\cdot}719$ and $\alpha = 81^\circ\ 33'$ on the theory of an abrupt transition. For gold in water ($\mu_0 = 1{\cdot}33$) these constants would lead to $\phi = 68^\circ\ 49'$ and $\beta = 41^\circ\ 30'$, whereas Conroy found $\phi = 67^\circ\ 39'$ and $\beta = 41^\circ\ 15'$. For gold in carbon bisulphide ($\mu_0 = 1{\cdot}63$) theory would give $\phi = 66^\circ\ 58'$ and $\beta = 41^\circ\ 46'$ instead of $\phi = 66^\circ\ 36'$ and $\beta = 41^\circ\ 41'$ as obtained by Conroy. With different colours for gold and also for silver the same discrepancy between theory and experiment would also be apparent, the differences being in nearly every case in the same direction, the theoretical results being too large. This discrepancy is just what the above discussion would lead us to expect, if there is a layer of transition between the two media.

Summary.

The chief results of the present investigation are the following:—

1. That in metallic reflection, if the transition from one medium to the other be abrupt, the Principal Incidence is always near the quasi-polarising angle, and is given very approximately by the formula

$$\sec\phi = M + M^{-1}(1 - \tfrac{1}{2}\cos 2\alpha).$$

2. That even when great care is taken to clean the surface of a metal the transition from it to the neighbouring medium is often gradual and not abrupt. This is in accordance with experimental and theoretical investigations on reflection from *transparent* substances such as glass and diamond.

3. That the influence of this layer on the ellipticity of the reflected light and on the difference of phase between light polarised perpendicularly and parallel to the plane of incidence extends over a wider range than in the case of transparent substances.

4. That the thickness of the layer is of about the same order of magnitude as with transparent media.

5. That the layer affects the position of the Principal Incidence considerably, and also influences the Principal Azimuth.

6. That, consequently, the deduction of the optical constants of a metal from observation of the Principal Incidence and Azimuth alone is liable to considerable error. [In the case of steel this method leads to $\mu = 2{\cdot}249$ and $a = 3{\cdot}257$ (see p. 216), while the wider theory yields $\mu = 2{\cdot}134$ and $a = 2{\cdot}906$.]

7. That four constants are required to describe the optical properties of a metallic reflector, two of them depending on the nature of the layer of transition.

8. That with these four constants a very satisfactory agreement exists between theory and experiment, as regards both the intensity of the reflected light and the difference of phase between the lights polarised perpendicularly and parallel to the plane of incidence.

The Relation Between the Osmotic Pressure and the Vapour Pressure in a Concentrated Solution.

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(Communicated by W. C. D. Whetham, F.R.S. Received November 11, 1905; received in Revised form January 3,—Read January 25, 1906.)

1. The relation between the vapour pressure and the osmotic pressure of a solution is often investigated by considering the equilibrium of a column of solution separated at the bottom from the pure solvent by a semi-permeable membrane, and placed in an atmosphere of vapour from the solvent. Now the hydrostatic equilibrium of the vapour column gives

$$\delta p = gs^{-1}\delta h,$$

where p is the vapour pressure of the pure solvent, g the acceleration due to gravity, h the height above the surface of the pure solvent, and s the specific volume of the vapour. Hence considering the equilibrium of the liquid column we get

$$\mathrm{P}+p-p' = \rho hg = \rho\int_{p'}^{p} s\,dp,$$

when P is the osmotic pressure, p' the vapour pressure of the solution, p that of the pure solvent, and where ρ is the effective mean density of the column of liquid.